Basic Notes on Finite Group Representations and Total Variation Distance

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1 Introduction

These notes collect some fundamental ideas in representation theory of finite (and, briefly, compact) groups, with a particular emphasis on the regular representation, character theory, and a standard lemma for bounding distances between probability measures on a group. They are intended for an upper-level undergraduate who has some familiarity with linear algebra, group theory, and basic concepts of analysis (like norms and metrics on finite-dimensional vector spaces).

We begin by introducing the notion of total variation distance for probability measures, prove a simple lemma relating it to the ℓ^1-norm, and then connect this to the idea of characters and irreducible representations in the abelian and non-abelian settings. We also highlight the role of the regular representation, orthogonality relations, and (in the finite setting) how one can diagonalize representations in the abelian case. Finally, we briefly touch on the analog for compact groups and the Peter–Weyl theorem.

2 Total Variation Distance on Probability Measures

Let G be a finite group. A *probability measure* P on G is a function $P : G \to [0,1]$ such that

$$\sum_{g\in G} P(g) \;=\; 1.$$

Often, we think of P as putting mass $P(g)$ at the point g.

Definition 2.1 (Total Variation Distance)**.** *Given two probability measures P and Q on a finite set G, the* total variation distance *between them is defined as*

$$\|P-Q\|_{\mathrm{TV}} \;=\; \max_{A\subseteq G}\bigl|P(A)-Q(A)\bigr| \;=\; \max_{A\subseteq G}\Bigl|\sum_{g\in A}[P(g)-Q(g)]\Bigr|.$$

Lemma 2.2. *For two probability measures P and Q on G,*

$$\|P-Q\|_{\mathrm{TV}} \;=\; \tfrac{1}{2}\|P-Q\|_1,$$

where $\|P-Q\|_1 = \sum_{g\in G}|P(g)-Q(g)|$.

Proof. Let $f(g) = P(g) - Q(g)$. Then

$$\|P-Q\|_{\mathrm{TV}} \;=\; \max_{A\subseteq G}\Bigl|\sum_{g\in A} f(g)\Bigr|.$$

A standard argument (often in basic probability theory) shows that $\max_{A\subseteq G}\bigl|\sum_{g\in A} f(g)\bigr| = \frac{1}{2}\sum_{g\in G}|f(g)|$. (The idea is to choose A to contain exactly those g for which $f(g)\geq 0$, maximizing the absolute sum in one direction or the other.) Thus

$$\|P-Q\|_{\mathrm{TV}} \;=\; \tfrac{1}{2}\sum_{g\in G}|f(g)| \;=\; \tfrac{1}{2}\|P-Q\|_1.$$

□

3 Representations of Finite Groups

We now turn to basic representation theory. Our main interest is in how finite groups act on vector spaces, and how we can use characters to glean information about these actions.

Definition 3.1 (Representation)**.** *Let G be a finite group. A (complex)* representation *of G is a group homomorphism*

$$\rho : G \;\to\; GL(V),$$

where V is a finite-dimensional complex vector space and $GL(V)$ is the group of invertible linear operators on V. Equivalently, it is a way for G to act on V by linear transformations.

Definition 3.2 (Irreducible Representation)**.** *A representation $\rho : G \to GL(V)$ is said to be* irreducible *(or* simple*) if V has no proper, non-zero, $\rho(G)$-invariant subspaces. In other words, the only subspaces of V invariant under the action of all $\rho(g)$ $(g \in G)$ are $\{0\}$ and V itself.*

Definition 3.3 (Character)**.** *Given a representation* $\rho : G \to GL(V)$*, its* character *is the function* $\chi_\rho : G \to \mathbb{C}$ *defined by*

$$\chi_\rho(g) \;=\; \operatorname{trace}\bigl(\rho(g)\bigr).$$

Two key facts (whose proofs can be found in standard texts) are:

1. Every finite-dimensional representation of a finite group *completely reduces* into a direct sum of irreducibles.

2. Characters form an orthonormal set with respect to a certain inner product (the *orthogonality relations*).

4 The Regular Representation

A central example in the representation theory of finite groups is the *regular representation.*

Definition 4.1 (Group Algebra $\mathbb{C}[G]$ and Regular Representation)**.** *The* group algebra $\mathbb{C}[G]$ *is the vector space over* $\mathbb{C}$ *with basis* $\{e_g : g \in G\}$*, whose dimension is* $|G|$*. An element of* $\mathbb{C}[G]$ *is of the form*

$$\sum_{g\in G} a_g e_g, \quad a_g \in \mathbb{C}.$$

There is a natural left regular action *of* G *on* $\mathbb{C}[G]$ *given by left multiplication:*

$$h \cdot e_g = e_{hg}, \tag{1}$$

extended linearly. This action defines the regular representation*, denoted* ρ_{reg}*.*

Proposition 4.2 (Character of the Regular Representation)**.** *Let* ρ_{reg} *be the regular representation of* G*. Then its character* χ_{reg} *satisfies:*

$$\chi_{\text{reg}}(g) \;=\; \begin{cases} |G|, & \text{if } g = e, \\ 0, & \text{if } g \neq e. \end{cases}$$

Sketch of Proof. Under $\rho_{\text{reg}}(g)$, the basis vector e_h is sent to e_{gh}. If $g \neq e$, then no basis vector is fixed, and the permutation of basis vectors has no fixed points. Hence its trace is 0. If $g = e$, the identity map fixes every basis vector e_h, so its trace is the full dimension, $|G|$. □

Theorem 4.3 (Decomposition of the Regular Representation)**.** *Every irreducible representation* ρ_i *of* G *appears in the regular representation* ρ_{reg} *with multiplicity equal to its dimension. Concretely,*

$$\rho_{\text{reg}} \;\cong\; \bigoplus_{i=1}^{r} \rho_i^{\oplus \dim(\rho_i)},$$

where $\rho_1, \rho_2, \ldots, \rho_r$ *are all the distinct irreducible representations of* G*.*

Corollary 4.4 (Orthogonality Relation for Characters)**.** *Let $\{\chi_i\}_{i=1}^r$ be the characters of the distinct irreducible representations of G. Then*

$$\sum_{i=1}^{r} \dim(\rho_i)\,\chi_i(g) \;=\; 0 \quad \textit{for all } g \neq e,$$

and equals $|G|$ when $g = e$. Equivalently, if you sum up $\dim(\rho_i)\,\chi_i$ over all irreps, you get the character of the regular representation.

Proof. This follows immediately from the decomposition of ρ_{reg} and the fact that $\chi_{\text{reg}}(g) = 0$ for $g \neq e$ and $|G|$ for $g = e$. □

5 The Abelian Case

If G is abelian, then every irreducible representation has dimension 1. Indeed, a standard theorem asserts:

Theorem 5.1. *If G is a finite abelian group, then it has $|G|$ distinct irreducible representations, each of which is one-dimensional. Equivalently, each irreducible character is a group homomorphism $\chi : G \to \mathbb{C}^\times$.*

Idea of Proof. Since G is abelian, every representation can be simultaneously diagonalized (in a suitable basis) by Schur's Lemma. A nontrivial block of dimension > 1 would yield non-commuting matrices if the representation were irreducible. Thus all irreps are 1-dimensional. □

Example 5.2. *If $G = \mathbb{Z}_n = \langle x \mid x^n = e \rangle$, then each irreducible character χ_k is given by*

$$\chi_k(x^m) \;=\; e^{2\pi i k m/n}, \quad k = 0, 1, \ldots, n-1.$$

Hence there are exactly n irreps, all of dimension 1.

5.1 Fourier Transform on a Finite Abelian Group

When G is abelian, the collection of all irreps (i.e., characters) $\widehat{G}$ itself forms a group under pointwise multiplication, often called the *dual group*. The map

$$f : G \to \mathbb{C} \;\;\mapsto\;\; \widehat{f} : \widehat{G} \to \mathbb{C},$$

where

$$\widehat{f}(\chi) \;=\; \sum_{g \in G} f(g)\,\overline{\chi(g)},$$

is the *Fourier transform* on G. This is a powerful tool for analyzing probability measures on abelian groups (e.g. random walks), relating the distribution to its character table.

6 Applications to Probability Measures and the Upper Bound Lemma

One important application (popularized by Persi Diaconis and others) is to bound how quickly a random walk on G converges to the uniform distribution in total variation distance. The key tool is to analyze the Fourier transform (or characters) of the distribution of the random walk.

Remark 6.1 (Upper Bound Lemma in Random Walks)**.** *There is a well-known* Upper Bound Lemma *(cf. Diaconis) stating that if P is a probability measure on G describing one step of a random walk, then*

$$\|P^{*t} - U\|_{\mathrm{TV}} \;\le\; \frac{1}{2} \sum_{\substack{\rho \text{ irreps} \\ \rho \neq \rho_{\mathrm{trivial}}}} \dim(\rho)\,\big|\lambda_\rho(P)\big|^t,$$

where P^{*t} is the t-fold convolution of P with itself, U is the uniform measure on G, and $\lambda_\rho(P)$ are certain Fourier coefficients *(eigenvalues) associated to ρ. This inequality encapsulates how fast the random walk's distribution mixes toward uniform.*

We will not go into the full proof here, but the essential idea is to decompose the *convolution operator* on $\mathbb{C}[G]$ into irreducible subrepresentations and then use the fact that each subrepresentation provides a distinct eigenvalue. The trivial representation contributes the uniform distribution in the limit, and all other irreps typically contract.

7 Brief Mention of the Compact Case: Peter–Weyl Theorem

For a compact (possibly infinite) group K, there is an analog of these results known as the *Peter–Weyl theorem*. It says that every (continuous) representation of a compact group on a finite-dimensional Hilbert space decomposes into irreducibles, and that the space $L^2(K)$ can be viewed as the (possibly infinite) direct sum of all irreps. In particular, for compact Lie groups such as SO(3) or SU(2), one obtains a beautiful theory of matrix coefficients and spherical harmonics.

In the finite case, the Peter–Weyl theorem reduces to the statement that $\mathbb{C}[G]$ decomposes into irreps exactly as we saw with the regular representation. For infinite compact groups, integrals replace sums, and one obtains the same type of orthogonality relations in an L^2-sense.

8 Summary

These notes illustrate how representation theory—especially the decomposition of the regular representation and the use of characters—can be used to understand fundamental questions about distributions on finite groups (like measuring distance in total variation). Key points include:

- The total variation distance on probability measures and its relationship to the ℓ^1-norm.
- The definition and decomposition of the regular representation, whose character is $|G|$ at the identity and 0 elsewhere.
- Orthogonality relations and how the sum of $\dim(\rho_i)\,\chi_i$ over all irreps ρ_i recovers the regular character.

- In the abelian case, all irreps are 1-dimensional, which greatly simplifies the character theory (and leads to the classical discrete Fourier transform).
- Applications to random walks on groups (the Upper Bound Lemma) and the analogy with the Peter–Weyl theorem in the compact case.

For more details, see standard references such as:

- Serre, *Linear Representations of Finite Groups.*
- Diaconis, *Group Representations in Probability and Statistics.*
- Fulton and Harris, *Representation Theory: A First Course.*